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EFFECTIVE COFINALITIES AND ADMISSIBILITY
IN E-RECURSION

by

E.R. Griffor and D. Normann^{*)}
Institute of Mathematics
University of Oslo

^{*)} The first author was partially supported by a grant from the National Science Foundation, USA.

Effective Cofinalities and Admissibility in E-Recursion

E.R. Griffor and D. Normann
Institute of Mathematics
University of Oslo

§ 0 Introduction.

E-Recursion was introduced by D. Normann [1978] as a natural generalization of normal Kleene recursion in objects of finite type. Unless otherwise stated the E-closed sets we shall consider shall be of the form $E(\alpha)$ for some $\alpha \in \text{OR}$.

In § 1 we introduce the $\text{RE} \wedge \text{co-RE}$ cofinality and show that the Σ_1 -admissibility of $E(\alpha)$ implies that its greatest cardinal has $\text{RE} \wedge \text{co-RE}$ cofinality ω . In addition we show that RE -cofinality ω does not imply admissibility.

Section § 2 is devoted to a dynamic proof of selection (i.e. $\gamma = \text{cf}^{E(\alpha)}(\alpha)$) then we have uniform selection over RE subsets of any $\delta < \gamma$ on $E(\alpha)$), which can therefore be relativized. This selection theorem thus has among its corollaries the consistency of the extended plus one hypothesis at the type three level with $\neg\text{CH}$.

Applications of the proof of selection given in § 2 are presented in § 3. We show that if γ is the cofinality of α in $E(\alpha)$, then the co-RE cofinality of γ is γ . The proof of this gives rise to an effective covering property, namely, any co-RE subset of γ can be covered by a REC set of the same order type. The final application makes clear the connection between selection and singularities. We show that for $\alpha < \beta$ such that $\text{cf}(\beta) \leq \alpha$

by a function f recursive in α, β and some $\delta < \alpha$, then $\text{cf}(\beta) \leq \alpha$ by some f recursive in α, β .

The last section (§ 4) treats the interplay between monotone inductive definitions and E -recursive set functions using methods from Girard's β -logic [198?], without introducing β -logic or its proof theory. If a $\Delta_0 \varphi(x, \cdot)$ always has a solution in Γ_x^∞ (the least fixed point of monotone inductive Γ over x), then the function giving that solution is E -recursive in x . As a corollary we have an elementary proof of a theorem of Van de Wiele [1981]:

If $F : V \rightarrow V$ is uniformly Σ_1 -definable and total over all admissible sets, then F is E -recursive. Outside of § 4, **RE**, **co-RE** etc. are the boldface notions.

§ 1. Effective Cofinalities.

Much attention has been given to various notions of definable cofinality, particularly in connection with priority arguments E -Recursion. We shall not attempt to give a complete picture and so the interested reader is directed to Griffor [1980], Sacks [1980] or Slaman [1981]. The first question we address here was asked by Sacks, namely, is there a cofinality condition on α which characterizes when $E(\alpha)$ is Σ_1 -admissible. The question was motivated by a result of Krousis that: if $E(\alpha) \models \text{cf}(\bar{\alpha}) = \omega$, then $E(\alpha)$ is Σ_1 -admissible. Thus an attractive conjecture was that: $E(\alpha)$ is Σ_1 -admissible if and only if $E(\alpha) \models \text{cf}(\bar{\alpha}) = \omega$. However, Slaman noticed that if γ is the least ordinal where $E(\gamma) \models \text{cf}(\bar{\gamma}) > \omega$, then $E(\gamma)$ is Σ_1 -admissible. If $E(\alpha)$ is Σ_1 -admissible Sacks [1980] showed that there is a divergent

computation without a Moschovakis witness in $E(\alpha)$. This witness induces an ω -sequence through $\bar{\alpha}$ and we will first analyse the level of definability of one such sequence.

Definition. Consider $E(\alpha)$, $\alpha \in \text{OR}$, and without loss of generality assume that α is the greatest cardinal in $E(\alpha)$. Define the RE join co-RE cofinality of α as:

$\text{RE} \wedge \text{co-RE-cf}(\alpha) = \text{least } \tau \leq \alpha \text{ such that there exists an } R \leq \alpha \text{ of order type } \tau \text{ unbounded in } \alpha \text{ and } R \text{ is } \text{RE} \wedge \text{co-RE, i.e. } R \text{ is the intersection of an RE and a co-RE set.}$

Theorem 1.0. Suppose $E(\alpha)$ is Σ_1 -admissible, then

$$\text{RE} \wedge \text{co-RE-cf}(\alpha) = \omega.$$

proof. As above we assume that α is the greatest cardinal in $E(\alpha)$ (which is L_κ for some $\kappa > \alpha$). If $e \in \omega$, $a \in E(\alpha)$, then associated with the computation tuple $\langle e, a \rangle$ is the tree of sub-computations $T_{\langle e, a \rangle}$ (which is recursive in $\langle e, a \rangle$ if $\{e\}(a) \downarrow$, but is in general only RE in $\langle e, a \rangle$). Assume that $E(\alpha)$ is Σ_1 -admissible.

By Sacks [1980] there exists an $e \in \omega$ and $a \in E(\alpha)$ such that $T_{\langle e, a \rangle}$ is not well-founded, but

$$L_\kappa \models 'T_{\langle e, a \rangle} \text{ is well-founded}'.$$

Claim 1. The leftmost path in $T_{\langle e, a \rangle}$ is in $\text{RE} \wedge \text{co-RE}$

proof. We say that σ is on the leftmost path if

- (i) $\sigma \in T_{\langle e, a \rangle}$ (RE)
- (ii) $\sigma \uparrow$ (co-RE)

(iii) If $\tau < \sigma$ in the lexicographical ordering and n is minimal such that $\tau(n) < \sigma(n)$, then $\tau(n+1) \downarrow$ (RE).

This proves claim 1.

Now assume that we have an effective coding of all finite sequences from α by α such that

$$\langle \sigma \wedge \tau \rangle > \langle \sigma \rangle, \text{ where } \tau \neq \langle \rangle.$$

Let $\langle \beta_1, \dots, \beta_n \rangle \in A$ if β_i is the index for the i^{th} sequence of the leftmost path through $T_{\langle e, a \rangle}$. Then A is the intersection of an RE set A_1 and a co-RE set A_2 .

Claim 2. A is unbounded in α .

proof. If A is bounded by $\lambda < \alpha$, then use standard properties of the Σ_1 -projectum on admissible ordinals to show that $A_1 \cap \lambda \in E(\alpha)$, $A_2 \cap \lambda \in E(\alpha)$ and so $A \in E(\alpha)$, which is impossible.

This completes the proof of the theorem.

Definition. With $E(\alpha)$ as above let

- (i) $\text{REC} - \text{cf}(\alpha) = \mu \tau \leq \alpha$ such that there exists REC $R \subseteq \alpha$ of order type τ unbounded in α ;
- (ii) $\text{RE} - \text{cf}(\alpha) = \mu \tau \leq \alpha$ such that there exists RE $R \subseteq \alpha$ of order type τ unbounded in α .

As one might expect the recursive cofinality is no stronger, on ordinals less than κ , than the cofinality in the sense of $E(\alpha)$.

Proposition 1.1. If $\gamma < \kappa$, then

$$\text{REC} - \text{cf}(\gamma) = \text{cf}^{\text{L}}_{\kappa}(\gamma).$$

proof \leq : let $f : \text{cf}^{\text{L}_\kappa}(\gamma) \rightarrow \gamma$, $f \in \text{L}_\kappa$ witness $\text{cf}^{\text{L}_\kappa}(\gamma)$ and without loss of generality we may assume that f is strictly increasing. Let $R = \text{im}(f)$, then R witnesses

$$\text{REC} - \text{cf}(\gamma) \leq \text{cf}^{\text{L}_\kappa}(\gamma).$$

\geq : let $R \subseteq \gamma$ witness the $\text{REC} - \text{cf}(\gamma) = \tau$, then $R \in \text{L}_\kappa$ by the bounding principle and the function $f : \tau \rightarrow \gamma$ given by: $\sigma < \tau$.

$$f(\sigma) = \sigma^{\text{th}} \text{ element of } R$$

is in L_κ and witnesses $\text{cf}^{\text{L}_\kappa}(\gamma) \leq \text{REC} - \text{cf}(\gamma)$.

Corollary 1.2. If $\text{REC} - \text{cf}(\alpha) = \omega$, then $E(\alpha)$ is Σ_1 -admissible.

proof. Use the proposition and the selection-theorem of Kirousis [1978] stating

$$E(\alpha) \models \text{cf}(\bar{\alpha}) = \omega \Rightarrow E(\alpha) \text{ is } \Sigma_1\text{-admissible.}$$

We shall see now that $\text{RE} - \text{cf}(\alpha) = \omega$ is not enough to guarantee admissibility.

Theorem 1.3. $\text{RE} - \text{cf}(\alpha) = \omega \not\Rightarrow E(\alpha)$ is Σ_1 -admissible.

proof. Begin with $E(\aleph_1)$ (which is not Σ_1 -admissible) and define the following κ_R -sequence:

$$\begin{aligned} \kappa_R(0) &= \kappa_R & ; \\ \kappa_R(n+1) &= \kappa_R^{\kappa_R(n)} & . \end{aligned} \quad \text{Now consider}$$

$\{x \mid x \in E(\aleph_1) \text{ and } x \leq_E \kappa_R(n) \text{ for some } n \in \omega\} = M$. Let \bar{M} be the Mostowski collapse of M , then \bar{M} is E -closed and satisfies the Moschovakis Phenomenon (use the MP in $E(\aleph_1)$ and the definition of κ_R) and \bar{M} is an E -closure of one of its elements.

But \bar{M} has an ω -sequence of κ_r 's. Let $\alpha = (\aleph_r)_{r \in \omega}$ and let

$$R = \{x < \alpha \mid x \text{ is the index for an ordinal } \beta \text{ such that } \beta = \kappa_r^a \text{ for some } a < \alpha\}.$$

R is RE and unbounded in α and clearly of order type ω . Thus \bar{M} is not Σ_1 -admissible, while over \bar{M} $\text{RE-cf}(\alpha) = \omega$, where $\alpha = (\aleph_r)_{r \in \omega}$.

§ 2. Dynamic Selection.

We shall give a dynamic proof of the following theorem:

Let α be the greatest cardinal in $E(\alpha)$ and let γ be the $E(\alpha)$ -cofinality of α . Then we have uniform selection for RE subsets of any $\delta < \gamma$.

As it stands, the theorem was proven by Kirousis [1978], but the 'dynamic' proof we shall give can be relativized, whereas Kirousis made use of a Skolem Hull - collapsing argument. A similar proof using a collapsing argument was given by Normann [1979] for the case $\gamma = \alpha$, i.e. α is a regular cardinal in $E(\alpha)$. We now give the dynamic proof.

Let δ be fixed as in the theorem and let f be a δ -sequence of computations. Let R be the Moschovakis [1967] sub-computation relation which is RE and, finally, let R_β denote the β^{th} approximation to R . The relation R is such that for a given computation, the set of immediate subcomputations can uniformly be indexed by a finite set or by α (the case of an

α -branching). In the case of composition we let the innermost computation be the leftmost one. If this one is convergent, then we know the other subcomputations.

Following Harrington-MacQueen [1976] we let

$$\min(f) = \inf\{\|f(y)\| : y < \delta\}, \text{ where}$$

$\|\cdot\|$ denotes the function giving the height of a computation, if convergent, and equals ∞ otherwise. If $\min(f) < \infty$, i.e. one of the $f(y)$'s is convergent, we shall show that $\min(f)$ is uniformly recursive in f for $f \in E(\alpha)$. The situation $\min(f) < \infty$ corresponds to the non-emptiness of the associated RE subset of δ and, thus, we have shown selection over δ .

The proof proceeds by transfinite induction on $\min(f)$. An application of the recursion theorem yields the required uniformity.

The relation $\min(f) = 0$ is recursive, so assume that $\min(f) > 0$ and that we have computed $\min(g)$ for all g such that $\min(g) \leq \min(f)$.

If $\min(f) > \beta$ (which is recursive in β) we let

$$g_\beta(y) = \text{leftmost subcomputation } z \text{ of } f(y) \text{ such that } \|z\| \geq \beta ;$$

and otherwise we let $g_\beta = f$. Clearly g_β is recursive in f, β and if $\min(f) > \beta$, then

$$\beta \leq \min(g_\beta) < \min(f).$$

Let τ be a recursive function defined by:

$$\tau(0) = 1 ;$$

$$\tau(\lambda) = \sup\{\tau(\beta) \mid \beta < \lambda\} \text{ if } \lambda \text{ is}$$

a limit ordinal;

$$\tau(\beta+1) = \min(g_{\tau(\beta)+1}).$$

Claim. $\tau(\alpha) \geq \min(f)$.

proof (Claim) Otherwise for each $\beta < \alpha$ let $h_\beta = g_{\tau(\beta)+1}$, then if $\beta_1 < \beta_2$, there is a $y < \delta$ such that

$$h_{\beta_1}(y) < h_{\beta_2}(y). \text{ Let } \beta_y = h_\beta(y),$$

then if for some y , $\{\beta_y : \beta < \alpha\}$ is unbounded, we have

$\|f(y)\| \leq \tau(\alpha)$, so this cannot be the case. Let $\beta_y^* = \sup\{\beta_y | \beta < \alpha\}$.

Since

$$\delta < \gamma = \text{cf}^{E(\alpha)}(\alpha), \text{ we have that}$$

$$\sigma = \sup\{\beta_y^* | y < \delta\} < \alpha. \text{ But for each } \beta < \alpha$$

there is one minimal y such that $(\beta+1)_y > \beta_y$. This gives a one-to-one map of α into $\delta \times \sigma$, which is impossible and gives the claim.

Since $\tau(\alpha)$ is recursive, we have computed $\min(f)$ from f giving the theorem.

Corollary 2.0. We have selection over $\gamma = \text{cf}^{E(\alpha)}(\alpha)$ if and only if we have selection over α .

proof. Selection over α clearly implies selection over γ . The other direction follows from the theorem and the dynamic proof of selection due to Sacks-Slaman (Theorem 2.8 in Slaman [1981]) which inspired this proof.

Now assume that $E(\alpha)$ is not Σ_1 -admissible and, hence, we do not have selection over α . The above corollary tells us we do

not have selection over γ , however the theorem tells us:

Corollary 2.1. Let $\delta < \gamma$, $C \subseteq \delta$ be RE, then $C \in E(\alpha)$.

proof. Since we have selection over δ , it follows that

$$\sup\{\kappa_0^y \mid y < \delta\} < \kappa \text{ and}$$

C can be defined this level in $E(\alpha)$.

Corollary 2.2. (Further Reflection) Let δ, C be as above, then

$$(a) \quad \kappa_0^{C, \delta} < \kappa_r^\delta;$$

(b) if $B \subseteq E(\alpha)$ is RE and $B(C)$ holds, then there exists a δ -recursive β such that $B(C_\delta)$ holds.

proof. immediate.

Corollary 2.3. Suppose $2^\omega = \kappa$, κ is a regular cardinal and there is a well-ordering of 2^ω of height κ recursive in ${}^4\mathbb{R}$ and a real. Then the extended plus one hypothesis is true at the type 3 level.

This last corollary was pointed out to us by T. Slaman. The extended plus-one hypothesis (for reals) states: if F is a normal type $n+2$ object and $n \geq 1$, then there exists a normal type 3 object G such that

$$\frac{1}{2} \text{sc}(G) = \frac{1}{2} \text{sc}(F), \text{ where}$$

$\frac{1}{2} \text{sc}(F)$ is the collection of sets of reals recursive in F and some real.

For background and further results on the extended plus-one hypothesis see Sacks [1977] or Slaman [1981].

§ 3. Applications: co-RE Cofinality, Effective Covering and Uniform Computation of Cofinality.

We turn first to an application of the above selection result which will yield a covering property for many co-RE sets 'preserving cofinality' and characterize what will call co-RE cofinality. Let α be an ordinal and consider again $E(\alpha) = L_\kappa$ for some $\kappa > \alpha$. Without loss of generality we assume α is the greatest cardinal in L_κ and we let $\gamma = \text{cf}^{L_\kappa}(\alpha)$.

Definition. Let $\beta \leq \kappa$ and define the co-RE cofinality of β by:

co-RE - cf(β) = least δ such that there is a co-RE subset A of β of order type δ and unbounded in β .

Lemma 3.0. $\text{co-RE} - \text{cf}(\alpha) = \text{co-RE} - \text{cf}(\gamma)$.

proof. Let $f: \gamma \rightarrow \alpha$ be increasing and witness that $\text{cf}^{L_\kappa}(\alpha) = \gamma$.

\leq : If $A \subseteq \gamma$ is co-RE and of order type δ then $A_f = \{f(y) \mid y \in A\}$ is the same order type through α . If A is unbounded in γ , then A_f is unbounded in α .

\geq : Let $A \subseteq \alpha$ be co-RE, unbounded and of order type δ . Let $y \in A^*$, if there exists $z \in [f(y), f(y+1)) \cap A$. The RE sets are closed under the quantifiers $\forall z \in u$, so the co-RE sets are closed under $\exists z \in u$. Thus A^* is co-RE and clearly unbounded in γ . In addition $\text{o.t.}(A^*) \leq \text{o.t.}(A)$.

We shall show that $\text{co-RE} - \text{cf}(\gamma) = \gamma$. By the above selection

theorem, $\beta < \gamma$ implies that the RE predicates are uniformly closed under $\exists y < \beta$ and, in addition, that

$$L_{\kappa} \cap WF(\beta) \in L_{\kappa}, \text{ where } WF(\beta)$$

denotes the set of well-founded relations as $\beta \times \beta$ (the latter cannot in general be relativized).

Theorem 3.1. $\text{co-RE} - \text{cf}(\gamma) = \gamma$.

proof. Let $A \subseteq \gamma$ be co-RE, cofinal in γ of order type β . Let A_{δ} be the δ^{th} approximation to A from the outside, i.e.

$$A_{\delta} = \{y \mid L_{\delta} \not\models y \notin A\}.$$

We will show that there is a recursive δ such that $\text{o.t.}(A) = \text{o.t.}(A_{\delta})$.

Let $y < \gamma$, then $\text{o.t.}(A \cap y) < \beta$ and by Further Reflection applied to $\complement A$, there is a δ recursive in y such that

$$\text{o.t.}(A_{\delta} \cap y) < \beta.$$

Using this we construct a recursive increasing function $g: \gamma \rightarrow \kappa$ such that

$$\forall y < \gamma \ (\text{o.t.}(A_{g(y)} \cap y) < \beta).$$

Let $\delta = \sup\{g(y) \mid y < \gamma\}$, then δ is recursive so let $C = A_{\delta}$. Thus C is recursive and $A \subseteq C$. If $\text{o.t.}(C) > \beta$, then there exists a $y < \gamma$ such that $\text{o.t.}(C \cap y) = \beta$. But $C \cap y \subseteq A_{g(y)} \cap y$ since $g(y) < \delta$. Since $\text{o.t.}(A_{g(y)} \cap y) < \beta$, we have a contradiction.

Corollary 3.2. (Covering Property) Any co-RE subset A of γ can be covered by a REC set of the same order type.

The corollary is proven in the proof of the theorem and we

used the ordinal β as a parameter. This lack of uniformity makes extension of the result in the corollary to ordinals other than γ difficult, however we offer:

Problem. Is there a bounded co-RE set that cannot be covered by a REC set of the same order type?

If L_κ is Σ_1 -admissible, then $\text{co-RE} - \text{cf}(\kappa) = \omega$ (recall that $L_\kappa = E(\alpha)$), but the converse is not true.

As far as the questions of section § 1 go these results show that

$\text{co-RE} - \text{cf}(\alpha) = \omega \Rightarrow E(\alpha)$ is Σ_1 -admissible, however

$E(\alpha)$ Σ_1 -admissible $\not\Rightarrow \text{co-RE} - \text{cf}(\alpha) = \omega$.

Together with the results of § 2 this shows that there is no natural cofinality-assumption that will characterize when $E(\alpha)$ is admissible, the best seems to be the one implicit in the lack of certain Moschovakis Witnesses.

Our next application makes clear the interplay between selection and singularities.

Theorem 3.3. Let $\alpha < \beta$ be ordinals such that $\text{cf}(\beta) \leq \alpha$ by some function f recursive in α, β and some $\delta < \alpha$. Then $\text{cf}(\beta) \leq \alpha$ by some function recursive in α, δ .

proof. let $g: \alpha \rightarrow \beta$ be a list of 'computation tuples' over β such that $(\exists \delta < \alpha)[g(\delta) \downarrow]$. The intuition here is that we attempt to carry out a search for the $\delta < \alpha$ in question and we either compute it effectively, and hence the witness to $\text{cf}(\beta) \leq \alpha$, or we don't and in so doing (not doing) obtain a witness to $\text{cf}(\beta) \leq \alpha$.

Let $\min(g) = \min\{\|g(\delta)\| \mid \delta < \alpha\}$.

By the selection theorem in section 2: if $E(\beta) \models \text{cf}(\beta) \geq \alpha$, we know that $\min(g)$ is computable by some recursive function $M(g)$. In general it is sufficient for $M(g)$ to be defined that $\min(g)$ exists. If $M(g) < \min(g)$ this means that we have

$$E_{M(g)+1}(\alpha) \models \text{cf}(\beta) \leq \alpha; \text{ where for } \gamma < \text{OR} \cap E(\alpha)$$

$E_\gamma(\alpha) = \{x \in E(\alpha) \mid x \text{ computed by a computation of height } < \gamma\}$. Now let $g(\delta)$ be an index for f recursive in δ, α, β witnessing that $\text{cf}(\beta) \leq \alpha$. Since $\min(g)$ exists we have that the selection algorithm $M(g)$ satisfies $M(g) \downarrow$.

If $\min(g) = M(g)$ we have computed the level at which the cofinality map is constructed. If $M(g) < \min(g)$, this is because we know at that ordinal that $\text{cf}(\beta) \leq \alpha$. Thus in both cases we can find from $M(g)$ an f collapsing the cofinality of β below $\alpha+1$.

If $L_\kappa = E(\alpha)$ then for all γ such that $\alpha < \gamma < \kappa$ we can find effectively in α, γ a map in L_κ witnessing

$$\bar{\bar{\gamma}}^L_\kappa = \bar{\bar{\alpha}}^L_\kappa.$$

The above theorem will enable us to do this in many more cases. Suppose L_κ is E -closed and has a greatest cardinal ($\text{gc}(\kappa)$).

Corollary 3.4. If $\gamma > \text{gc}(\kappa)$, let f_γ be the least (in the sense of $<_L$) collapse of γ to $\text{gc}(\kappa)$. If for some $\alpha, \gamma_0 < \kappa$ we have that

$$(*) \quad (\forall \gamma > \gamma_0)(\exists z < \text{gc}(\kappa))[f_\gamma \leq_E \alpha, \gamma_0, \text{gc}(\kappa), \gamma, z],$$

then the function $\gamma \rightarrow f_\gamma$ is uniformly computable in $\gamma_0, a, gc(\kappa)$ and a $gc(\kappa)$ -enumeration of γ_0 .

proof. We proceed by induction on $\gamma \geq \gamma_0$.

$\gamma = \gamma_0$ is trivial. If $\gamma > \gamma_0$, let a_γ be so large that all $\gamma' < \gamma$ are collapsed to $gc(\kappa)$ by level α_γ . Let $\alpha \geq \alpha_\gamma$ such that:

if $L_{\alpha_\gamma} \models \bar{\gamma} > gc(\kappa)$, then

$L_\alpha \models \gamma = (gc(\kappa))^+$, where

τ^+ is the successor cardinal of τ . By the theorem there is an α recursive in $\gamma, a, \gamma_0, gc(\kappa)$ and the collapse of γ_0 such that

$L_\alpha \models cf(\gamma) \leq gc(\kappa)$.

But a successor cardinal is regular, so this singularity will demonstrate that $\bar{\gamma} = gc(\kappa)$ and the collapsing map can be computed.

Corollary 3.4 can be used to show that under (*) we have

Corollary 3.5. Let L_κ be E-closed and let $\alpha = gc(L_\kappa)$. Assume that $L_\kappa \models^{(*)}$. Then the following are equivalent

(i) L_κ is RE in an element of L_κ

(ii) Both $L_\kappa \cap (\alpha)$ and κ are RE in an element of L_κ .

Remark. Using forcing-methods of Sacks [198?] we may show that if * holds, then L_κ is not R.E.

§ 4. E-Recursive Functions and Inductive Definability.

In this section we shall give a treatment of monotone inductive definitions using methods from Girard's β -logic [198?], but without introducing β -logic and its proof theory. Masseron [1980] has used the proof theory of β -logic to show that every total ω_1^{CK} -recursive function on ω_1^{CK} is dominated by a primitive recursive dilator on infinite arguments. As a corollary we give a proof of Van de Wiele's theorem:

If $F:V \rightarrow V$ is total uniformly Σ_1 -definable over every admissible set, then F is E-recursive.

The converse for E-recursive functions (lightface) is immediate. Slaman has given an alternate proof, but his proof uses the theory of reflection in E-recursion, whereas we will require only familiarity with the generating schemata of E-recursion.

Like the completeness theorem for β -logic this proof is based on the Henkin-type construction of term models, otherwise the proof is elementary. For each set x let Γ_x be a uniformly $\Delta_0(x)$ positive inductive definition on x . Let \leq_x denote the stage comparison relation on x . The following lemma is valid for monotone inductive definitions in general.

Lemma 4.0. Let $Y \subseteq x$, \leq be a relation on y such that

(i) $\Gamma(Y) = Y$; and

(ii) for each $y \in Y$

$\{y' \mid y' \leq y\} = \Gamma(\{y' \mid y' < y\})$, then

$\Gamma_x^\infty \leq Y$ and \leq_x is the well-founded initial segment of \leq (Γ_x^∞ is the least fixed-point of Γ_x).

For each x , let τ_x be the closure ordinal of Γ_x and let φ be a Δ_0 -formula such that

$$\forall x \exists \gamma < \tau_x \varphi(x, \Gamma_x^{\gamma+1}).$$

Theorem 4.1. There is an E-recursive function G such that

$$\forall \alpha \forall x (\text{rank}(x) \leq \alpha \Rightarrow \exists \gamma \leq \min(G(\alpha), \tau_x) \varphi(x, \Gamma_x^{\gamma+1}));$$

Definition. Let $T = T_{\Gamma, \varphi}$ be the following first order theory:

unary predicates $\underline{x}, \underline{y}, \underline{ON}$
 binary predicates \underline{P} (for \leq_x) and $\underline{\in}$
 unary function R (for rank)
 constants $\underline{c_0}, \underline{c_1}, \dots$

Take standard axioms like regularity, extensionality, etc.
 together with:

- (i) $\underline{y} = \Gamma(\underline{y})$;
- (ii) $\varphi(x, \{y | \underline{P}(y, \underline{c_0})\}) \rightarrow \forall z \in \underline{y} (\varphi(x, \{y | \underline{P}(y, z)\}) \rightarrow \underline{P}(\underline{c_0}, z))$;
- (iii) $\underline{P}(\underline{c_{i+1}}, \underline{c_i}) \wedge \neg \underline{P}(\underline{c_i}, \underline{c_{i+1}})$; and
- (iv) $\forall z \in \underline{y} (\{y | \underline{P}(y, z)\} = \Gamma(\{y | \underline{P}(y, z) \wedge \neg \underline{P}(z, y)\}))$.

Definition (a) Let T_n denote the part of T that does not contain any $\underline{c_i}$ for $i \geq n$;

(b) Let T^*, T_n^* denote the respective Henkin-extensions ;

(c) Let $\underline{e_0}, \underline{e_1}, \dots$ be a recursive enumeration of the terms of T^* such that $\forall i (\underline{e_i} \in T_i^*)$.

Now if $f: \mathbb{N} \rightarrow \text{ON}$, let T^f be T^* extended with the following axioms:

$$\{R(\underline{e}_i) \leq R(\underline{e}_j) \mid f(i) \leq f(j)\}.$$

Lemma 4.2. Let $f: \mathbb{N} \rightarrow \text{ON}$ and T^f be as above, then T^f is inconsistent.

proof. Assume T^f is consistent for a contradiction and let \overline{T}^f denote a consistent completion of T^f . The term model for \overline{T}^f will then be a model of T and since the rank-relation is well-founded, the model will be isomorphic to a set z where \underline{x} is interpreted as a subset of z . Let $\gamma < \tau_{\underline{x}}$ be such that $\varphi(\underline{x}, \Gamma_{\underline{x}}^{\gamma+1})$. By lemma the interpretation \underline{c}_0 of c_0 must be in $\Gamma_{\underline{x}}^{\infty}$ and have rank $\leq \gamma+1$. But then interpretations of \underline{c}_i will form an \leq -infinite descending sequence, which is absurd.

If σ is a finite sequence of ordinals we define T^σ as an extension of $T_{\text{lh}(\sigma)}^*$ as before. Thus we have

$$\forall f: \mathbb{N} \rightarrow \text{OR} \exists r \in \mathbb{N} [T^{\overline{f(n)}} \text{ is inconsistent}].$$

Definition. Let σ be a sequence of ordinals of length n , then we say σ is good if we cannot prove a contradiction from T^σ using a proof of length $\leq n$ and at most the n first axioms of T^σ (in some uniform enumeration of T^f 's).

For $\alpha \in \text{OR}$ we let

$$S_\alpha = \{\sigma \mid \sigma \text{ is good and } \forall i < \text{lh}(\sigma) (\sigma(i) < \alpha)\}$$

and set $G(\alpha) = \text{height of } S_\alpha$. Then G is E-recursive since we can uniformly compute the height of any well-founded relation in E-recursion.

Lemma 4.3. Let $\text{rank}(x) \leq \alpha$, then we can find $\gamma \leq G(\alpha)$ such that $\varphi(x, \Gamma_x^{\gamma+1})$ holds.

proof. Fix x and let γ be minimal such that $\varphi(x, \Gamma_x^{\gamma+1})$ and choose $y \in \Gamma_x^{\gamma+1} - \Gamma_x^\gamma$. Let p denote the ordinal norm function on Γ_x^∞ induced by Γ_x . Then we have $p(y) = \gamma$. Assume that y_0, \dots, y_{n-1} is a sequence from Γ_x^∞ such that $y_0 = y$ and $p(y_i) < p(y_{i-1})$ for $1 \leq i < n$.

We shall construct a model for T_n using $\text{TC}(x)$ as the domain, x for \underline{x} , Γ_x^∞ for \underline{Y} , \leq_x for \underline{P} and y_0, \dots, y_{n-1} for $\underline{c}_0, \dots, \underline{c}_{n-1}$. This model can be extended to a model for T_n^* since T_n^* is a conservative extension of T_n and we do not change the domain. For $i < n$ let $\sigma(i) = \text{rank}(e_i)$ (e_i is the interpretation of \underline{e}_i). Note that if we extend \vec{g} in a consistent way, then we may extend σ (i.e. we cannot choose σ such that it is inconsistent with the construction based on extensions of \vec{y}).

If $\alpha = \text{rank}(x)$, then $\text{rank}(e_i) < \alpha$ by our choice of domain as $\text{TC}(x)$ and so $\sigma \in S_\alpha$. By induction on $p(y_{n-1})$ we can show that $p(y_{n-1}) \leq \|\sigma\|_{S_\alpha}$. The induction is trivial by the above remark on the consistency-considerations and, hence, the lemma follows. The theorem follows from the lemma.

Remark. The theory T in the proof asserts that x is a relation on a transitive set y ; $\langle Y, P \rangle$ is the prewellordering induced by Γ over x and there is no $z \in \Gamma_x^\infty$ satisfying φ . If T' is a primitive recursive theory in the language of set theory, then the same proof gives:

Corollary 4.4. Let Γ , φ and τ_x be as above. If

$\forall x (x \models T' \Rightarrow \exists \gamma < \tau_x \varphi(x, \Gamma_x^{\gamma+1}))$ then there is an E-recursive function G such that

$$\forall x (x \models T' \Rightarrow \exists \gamma < \min\{\tau_x, G(\text{rank}(x))\} \varphi(x, \Gamma_x^{\gamma+1}))$$

Examples of such theories are:

- (i) x is transitive, infinite and closed under finite subsets;
- (ii) x is rudimentarily closed.

Now if x is transitive, infinite and closed under finite subsets, then we have a notation system for the next admissible $(\text{HYP}(x))$ and that notation system is defined by a monotone inductive definition. If $\exists y \in \text{HYP}(x) \varphi(x, y)$, then there is a Δ_0 formula φ' such that $\varphi'(x, \Gamma_x^\gamma)$ for the least γ such that $\exists y \in L_\gamma[x] \varphi(x, y)$ where Γ defines that notation system.

Using this we have proven the following theorem of J. Van de Wiele:

Corollary 4.5. (Van de Wiele) Let $F: V \rightarrow V$ be uniformly Σ_1 -definable and total over all admissible sets, then F is E-recursive.

proof follows immediately from the theorem and the above remarks on the inductive generation of $\text{HYP}(x)$.

Note that we actually show that F is computable in a weaker system than E-recursion, since we use elementary functions together with the operator which computes the height of a well-founded relation.

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